Superenergy and Supermomentum of Gödel Universes

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Abstract

We review the canonical superenergy tensor and the canonical angular supermomentum tensors in general relativity and calculate them for space-time homogeneous Gödel universes to show that both of these tensors do not, in general, vanish. We consider both an original dust-filled pressureless acausal Gödel model of 1949 and a scalar-field-filled causal Gödel model of Rebouças and Tiomno. For the acausal model, the non-vanishing components of superenergy of matter are different from those of gravitation. The angular supermomentum tensors of matter and gravitation do not vanish either which simply reflects the fact that Gödel universe rotates. However, the axial (totally antisymmetric) and vectorial parts of supermomentum tensors vanish. It is interesting that superenergetic quantities are *sensitive* to causality in a way that superenergy density ${}_gS_{00}$ of gravitation in the acausal model is *positive*, while superenergy density ${}_gS_{00}$ in the causal model is *negative*. That means superenergetic quantities might serve as criterion of causality in cosmology and prove useful.

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I. INTRODUCTION

In this paper we calculate the canonical superenergy tensor and the canonical angular supermomentum tensor for homogeneous spacetimes of the form first investigated in general relativity by Gödel in 1949 [1,2]. Since we can see galaxies are now rotating there have been suggestions that their rotation is primordial and originated from a global rotation of an early Gödel spacetime [3,4]. The Gödel's solutions attract considerable interest because they describe rotating universes that possess the completely unexpected property of closed timelike curves (CTCs). However, generalized Gödel models which do not contain CTCs have been found in general relativity in the presence of massless scalar fields [5], and in gravity theories derived from an action containing terms quadratic in the Ricci curvature invariants [6], in five-dimensional gravity theories [7], or in string-inspired gravity theories [8]. We study both causal and non-causal Gödel universes and find their superenergetic properties.

The paper is organizing as follows. In Section II, after giving some remarks about energy-momentum in general relativity, we present our definition of the superenergy tensors for gravitation and for matter. In Section III we introduce the angular supermomentum tensors and give some hints about the application of these superenergy and angular supermomentum tensors in general relativity. In Section IV we present a short review of some other approaches to the problem of energy and superenergy in general relativity. Section V is devoted to the calculation of superenergetic quantities for the acausal Gödel spacetime. In Section VI the superenergetic quantities for a generalized causal Gödel-Rebouças-Tiomno spacetime are given. In Section VII we present our conclusions and in the Appendix we give some useful formulas (for instance Bel-Robinson tensor components) to calculate superenergy and angular supermomentum for the models under consideration.

II. SUPERENERGY TENSORS

In general relativity the gravitational field which is given by Levi-Civita connection $\Gamma^i_{kl} = \Gamma^i_{lk}$ $(i, j, a, b \ldots = 0, 1, 2, 3)$ [1] *does not possess* an energy-momentum tensor. The only objects which can be defined are *energy-momentum pseudotensors* and this is a consequence of the Einstein Equivalence Principle. Old and new investigations (see, eg. [9]) have shown that the best solution to the energy-momentum problem in standard general relativity (without supplementary objects like distinguished tetrad fields, an auxilliary metric or an arbitrary vector field) seems to be the application of the Einstein canonical energy-momentum pseudotensor ${}_Et_i{}^k$ [10,11] and the canonical double-index energy-momentum complex [10,11]

$$ {}_EK_i{}^k = \sqrt{|g|}(T_i{}^k +_E t_i{}^k), \tag{2.1} $$

where T^{ik} is a symmetric energy-momentum tensor of matter which appears on the r.h.s. of the Einstein equations and $|g|$ is the determinant of the metric tensor.

The Einstein equations can be rewritten in the form [11]

$$ \sqrt{|g|}(T_i{}^k +_E t_i{}^k) = {}_FU_i{}^{kl}{}_{,l}, \tag{2.2} $$

where (in holonomic coordinates)

$$ {}_FU_i{}^{kl} = (-)_FU_i{}^{lk} = \alpha\frac{g_{ia}}{\sqrt{|g|}}[(-g)(g^{ka}g^{lb} - g^{la}g^{kb})]_{,b} \tag{2.3} $$

are Freud's superpotentials while

$$ \begin{aligned} {}_Et_i{}^k = \alpha\Big\{&\delta^k_i g^{ms}(\Gamma^l_{mr}\Gamma^r_{sl} - \Gamma^r_{ms}\Gamma^l_{rl}) \\ &+ g^{ms}{}_{,i}[\Gamma^k_{ms} - \frac{1}{2}(\Gamma^k_{tp}g^{tp} - \Gamma^l_{tl}g^{kt})g_{ms} - \frac{1}{2}(\delta^k_s\Gamma^l_{ml} + \delta^k_m\Gamma^l_{sl})]\Big\}, \end{aligned} \tag{2.4} $$

is the Einstein's gravitational energy-momentum pseudotensor. Here we have taken $(c = G = 1)$

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[1]Levi-Civita connection is symmetric in holonomic coordinates.

$$\alpha = \frac{1}{16\pi}. \tag{2.5}$$

Since ${}_E t_i^{\ k}$ can also be obtained from Einstein Lagrangian of the gravitational field as a canonical object (see e.g. [10]), then it is usually called the Einstein *canonical* energy-momentum pseudotensor for the gravitational field. On the other hand, ${}_E K_i^{\ k}$ is called the Einstein *canonical* energy-momentum complex for matter and gravitation. From (2.2) one gets the local, differential energy-momentum conservation laws

$$[\sqrt{|g|}(T_i^{\ k} +_E t_i^{\ k})]_{,k} = 0 \tag{2.6}$$

and the integral conservation laws (Synge's conservation laws)

$$\oint\limits_{\partial\Omega} \sqrt{|g|}(T_i^{\ k} +_E t_i^{\ k}) d\sigma_k = 0, \tag{2.7}$$

where $\partial\Omega$ is a boundary of a four-dimensional, compact domain Ω, and $d\sigma_k$ is the three-dimensional volume element [10].

In order to fill the gap for an energy-momentum tensor in general relativity, one can introduce the *canonical superenergy tensor* (and also other superenergy tensors) which was done in series of papers [12–15][2]. The idea of superenergy (originally introduced for the gravitational field) is quite universal in the sense that for any physical field with an energy-momentum tensor or pseudotensor constructed of Γ^i_{kl} one can *always* build up the corresponding superenergy tensor.

The general definition of the superenergy tensor $S_a^{\ b}(P)$ which can be applied to an arbitrary gravitational as well as matter field is

$$S_{(a)}^{\ \ (b)}(P) = S_a^{\ b} := \lim_{\Omega \to P} \frac{\int\limits_{\Omega} \left[T_{(a)}^{\ \ (b)}(y) - T_{(a)}^{\ \ (b)}(P) \right] d\Omega}{1/2 \int\limits_{\Omega} \sigma(P; y) d\Omega}, \tag{2.8}$$

where

[2]In this paper we use the signature $(-+++)$ despite $(+---)$ used in [12–15].

$$T_{(a)}{}^{(b)}(y) := T_i{}^k(y)e^i_{(a)}(y)e^{(b)}_k(y),$$
$$T_{(a)}{}^{(b)}(P) := T_i{}^k(P)e^i_{(a)}(P)e^{(b)}_k(P) = T_a{}^b(P) \tag{2.9}$$

are the tetrad components of a tensor or a pseudotensor field $T_i{}^k(y)$ which describe an energy-momentum, y is the collection of normal coordinates **NC(P)** at a given point **P**, $\sigma(P,y)$ is the world-function, $e^i_{(a)}(y)$, $e^{(b)}_k(y)$ denote an orthonormal tetrad field and its dual, respectively, $e^i_{(a)}(P) = \delta^i_a$, $e^{(a)}_k(P) = \delta^a_k$, $e^i_{(a)}(y)e^{(b)}_i(y) = \delta^b_a$, and they are paralell propagated along geodesics through **P**. For a sufficiently small domain Ω which surrounds **P** we require

$$\int\limits_{\Omega} y^i d\Omega = 0, \quad \int\limits_{\Omega} y^i y^k d\Omega = \delta^{ik} M, \tag{2.10}$$

where

$$M = \int\limits_{\Omega}(y^0)^2 d\Omega = \int\limits_{\Omega}(y^1)^2 d\Omega = \int\limits_{\Omega}(y^2)^2 d\Omega = \int\limits_{\Omega}(y^3)^2 d\Omega \tag{2.11}$$

is a common value of the moments of inertia of the domain Ω with respect to the subspaces $y^i = 0$. The procedure of an "averaging" of the energy-momentum given in (2.8) is an amended version of the procedure proposed by Pirani [16].

Let us choose Ω as a small ball defined by

$$(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \leq R^2, \tag{2.12}$$

which can be described in a covariant way in terms of the auxiliary positive-definite metric $h^{ik} := 2v^i v^k + g^{ik}$, where v^i are the components of the four-velocity vector of an observer **O** at rest at **P**. As for the world function we choose [17]

$$\sigma(P;y) \doteq \frac{1}{2}(-(y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2) \tag{2.13}$$

and $\doteq$ means it is valid only in some special coordinates. The world function can covariantly be defined by the eikonal-like equation

$$g^{ik}\partial_i\sigma\partial_k\sigma = 2\sigma, \tag{2.14}$$

with conditions: $\sigma(P,P)=0, \quad \partial_i\sigma(P,P)=0$. The equation (2.14) allows to rewrite (2.12) by

$$h^{ik}\partial_i\sigma\partial_k\sigma \leq R^2. \tag{2.15}$$

Since at **P** the tetrad and normal components are equal, from now on we will write the components of any quantity at **P** without (tetrad) brackets, e.g., $S_a{}^b(P)$ instead of $S_{(a)}{}^{(b)}(P)$ and so on.

Let us now make the following expansion for the energy-momentum tensor of matter [18]

$$\begin{aligned} T_i^k(y) &= \hat{T}_i^k + \nabla_l \hat{T}_i^k y^l + 1/2\hat{T}_i^k{}_{,lm} y^l y^m + R_3 \\ &= \hat{T}_i^k + \nabla_l \hat{T}_i^k y^l + 1/2\Big[\nabla_{(l}\nabla_{m)}\hat{T}_i^k \\ &- 1/3\hat{R}^c{}_{(l|i|m)}\hat{T}_c^k + 1/3\hat{R}^k{}_{(l|c|m)}\hat{T}_i^c\Big] y^l y^m + R_3, \end{aligned} \tag{2.16}$$

$$e^i_{(a)}(y) = \hat{e}^i_{(a)} + 1/6\hat{R}^i{}_{lkm}\hat{e}^k_{(a)} y^l y^m + R_3, \tag{2.17}$$

$$e^{(b)}_k(y) = \hat{e}^{(b)}_k - 1/6\hat{R}^p{}_{lkm}\hat{e}^{(b)}_p y^l y^m + R_3, \tag{2.18}$$

which gives (2.8) in the form

$${}_mS_a{}^b(P) = \lim_{\Omega\to P}\frac{\int\limits_\Omega(\nabla_l\hat{T}^b_a y^l + 1/2\nabla_{(l}\nabla_{m)}\hat{T}^b_a y^l y^m + THO)d\Omega}{1/2\int\limits_\Omega \sigma(P;y)d\Omega}. \tag{2.19}$$

THO means the terms of higher order in the expansion of the differences $T^{(b)}_{(a)}(y) - T^{(b)}_{(a)}(P) = T^{(b)}_{(a)}(y) - \hat{T}^b_a$, R_3 is the remainder of the third order, ∇ denotes covariant differentiation, and a hat denotes the value of an object at **P**.

The first and THO terms in the numerator of (2.19) do not contribute to ${}_mS^b_a(P)$. Hence, we finally get from (2.19)

$${}_mS^b_a(P) = \delta^{lm}\nabla_{(l}\nabla_{m)}T^b_a. \tag{2.20}$$

By introducing the four-velocity $v^l = \delta^l_0, \; v^l v_l = -1$ of an observer **O** at rest at **P** and the local metric $\hat{g}^{ab} \doteq \eta^{ab}$ (η^{ab} - Minkowski metric), one can write (2.20) in a covariant way as

$$ {}_{m}S_{a}{}^{b}(P;v^l) = (2v^l v^m + g^{lm})\nabla_{(l}\nabla_{m)}T_{a}{}^{b}. \tag{2.21}$$

This is the matter superenergy tensor ${}_{m}S_{a}^{b}(P;v^l)$ and it is symmetric. Of course, as a result of Pirani's "averaging" it does not satisfy any local conservation laws in general relativity. However, it satisfies trivial[3] conservation laws in special relativity.

Now let us take the gravitational field and make the expansion

$$\begin{aligned} {}_{E}t_{i}{}^{k}(y) = \frac{\alpha}{9}[&\hat{B}^{k}{}_{ilm} + \hat{P}^{k}{}_{ilm} - 1/2\delta_i^k \hat{R}^{abc}{}_{l}(\hat{R}_{abcm} + \hat{R}_{acbm}) + 2\delta_i^k \hat{R}_{(l|g}\hat{R}^{g}{}_{|m)} \\ &- 3\hat{R}_{i(l|}\hat{R}^{k}{}_{|m)} + 2\hat{R}^{k}{}_{(gi)(l|}\hat{R}^{g}{}_{|m)}]y^l y^m + R_3. \end{aligned} \tag{2.22}$$

This expansion (2.22) with the help of (2.17)-(2.18) gives the *canonical superenergy tensor for the gravitational field*

$$ {}_{g}S_{a}{}^{b}(P;v^l) = (2v^l v^m + g^{lm})W_{a}{}^{b}{}_{lm}, \tag{2.23}$$

where

$$\begin{aligned} W_{a}{}^{b}{}_{lm} = \frac{2\alpha}{9}[&B^{b}{}_{alm} + P^{b}{}_{alm} \\ &- \frac{1}{2}\delta_a^b R^{ijk}{}_{m}(R_{ijkl} + R_{ikjl}) + 2\delta_a^b R_{(l|g}R^{g}{}_{|m)} \\ &- 3R_{a(l|}R^{b}{}_{|m)} + 2R^{b}{}_{(ag)(l|}R^{g}{}_{|m)}], \end{aligned} \tag{2.24}$$

and

$$ B^{b}{}_{alm} := 2R^{bik}{}_{(l|}R_{aik|m)} - \frac{1}{2}\delta_a^b R^{ijk}{}_{l}R_{ijkm}, \tag{2.25}$$

is the *Bel–Robinson tensor*, while

$$ P^{b}{}_{alm} := 2R^{bik}{}_{(l|}R_{aki|m)} - \frac{1}{2}\delta_a^b R^{ijk}{}_{l}R_{ikjm}. \tag{2.26}$$

In vacuum, ${}_{g}S_{a}{}^{b}(P;v^l)$ reduces to a simpler form

[3]They are trivial because the integral superenergetic quantities for a closed system in special relativity vanish.

$$
{}_gS_a{}^b(P;v^l) = \frac{8\alpha}{9}(2v^lv^m + g^{lm})[R^{b(ik)}{}_{(l|}R_{aik|m)} - 1/2\delta^b_a R^{i(kp)}{}_{(l|}R_{ikp|m)}], \tag{2.27}
$$

which is also symmetric and the quadratic form ${}_gS_{ab}(P;v^l)v^av^b$ is positive-definite. The canonical superenergy tensor ${}_gS_a{}^b(P;v^l)$ is a tensor, contrary to the fact that the components ${}_Et_i{}^k$ do not form any geometric object. This is a consequence of the specific properties of the normal coordinates.

Our main conjecture is that ${}_gS_a{}^b(P;v^l)$ should be taken as a *substitute* for the energy-momentum tensor for the gravitational field. Its advantage is that it contains the Bel-Robinson tensor, which is a conserved quantity in vacuum. Unfortunately, superenergy tensors do not fulfill any conservation law. The components of ${}_gS_a{}^b(P;v^l)$ and ${}_mS_a{}^b(P;v^l)$ have the dimension which can be written down as: [the dimension of the components of an energy-momentum tensor (or pseudotensor)] $\times m^{-2}$. This might be considered to have a straightforward physical interpretation - namely: the energy-momentum tensor (or pseudotensor) is the *flux* of the appropriate superenergy tensor. Following Bel and Robinson, we call these tensors the *(canonical) superenergy tensors*.

III. SUPERMOMENTUM TENSORS

The idea to extend the notion of superenergy onto the angular momentum has also been proposed [14].

As it is known, the notion of an angular momentum in general relativity is much more complicated and obscure than the notion of an energy-momentum (see e.g. [19]). Even the matter field *does not possess* an angular momentum tensor because, in general, the *radius vector cannot be defined.* Moreover, one has serious difficulties with an invariant definition of the angular momentum in an asymptotically flat spacetime (at null or spatial infinity) [19], and with convergence of the resulting global angular momentum integrals in radiative spaces. However, for a closed system, i.e., in the case of an insular and nonradiating system [11], one can correctly define a global angular momentum, for example, by using Landau-Lifschitz [10] or Bergmann-Thomson [20] angular momentum complex. In an arbitrary asymptotically

flat spacetime one can construct a reasonable formula which gives temporal changes of the angular momentum [21].

The canonical angular supermomentum tensors introduced in [14] have much better geometric properties (they are tensors) than the angular momentum complexes from which they were obtained and they lead to global integrals which have *better convergence* (at least one order better in $0(r^{-n})$) than the corresponding global angular momenta integrals. It means that the global angular supermomenta can also be defined in the cases where the (global) angular momentum cannot be defined at all, i.e., when the suitable angular momentum integrals are divergent.

The canonical angular supermomentum tensors, analogous to the case of the canonical superenergy tensors, can be considered as substitutes for the angular momentum tensors of matter and gravitation which do not appear in general relativity. The constructive definition of these tensors, in analogy to the definition of the canonical superenergy tensors, is as follows. In normal coordinates **NC(P)** we define

$$S^{(a)(b)(c)}(P) = S^{abc}(P) := \lim_{\Omega \to P} \frac{\int\limits_{\Omega} [M^{(a)(b)(c)}(y) - M^{(a)(b)(c)}(P)] d\Omega}{1/2 \int\limits_{\Omega} \sigma(P;y) d\Omega}, \tag{3.1}$$

where

$$M^{(a)(b)(c)}(y) := M^{ikl}(y) e_i^{(a)}(y) e_k^{(b)}(y) e_l^{(c)}(y), \tag{3.2}$$

$$\begin{aligned} M^{(a)(b)(c)}(P) &:= M^{ikl}(P) e_i^{(a)}(P) e_k^{(b)}(P) e_l^{(c)}(P) = M^{ikl}(P) \delta_i^a \delta_k^b \delta_l^c \\ &= M^{abc}(P) \end{aligned} \tag{3.3}$$

are the physical (or tetrad) components of the field $M^{ikl}(y) = (-)M^{kil}(y)$ which describe angular momentum densities. As in (2.17)-(2.18), $e^i_{(a)}(y)$, $e_k^{(b)}(y)$ denote orthonormal bases such that $e^i_{(a)}(P) = \delta^i_a$ and its dual are parallel propagated along geodesics through **P** and Ω is a sufficiently small ball with centre at **P**. At **P** the tetrad and normal components of an object are equal. We apply this again (cf. Section II) and omit tetrad brackets for indices of

any quantity attached to the point $\mathbf{P}$; for example, we write $S^{abc}(P)$ instead of $S^{(a)(b)(c)}(P)$ and so on.

For matter as $M^{ikl}(y)$ we take

$$ {}_mM^{ikl}(y) = \sqrt{|g|}(y^iT^{kl} - y^kT^{il}), \tag{3.4} $$

where T^{ik} are the components of a symmetric energy-momentum tensor of matter and y^i denote the normal coordinates. The formula (3.4) gives the total angular momentum densities, orbital and spinorial because the dynamical energy-momentum tensor of matter T^{ik} comes from the canonical energy-momentum tensor by using the Belinfante-Rosenfeld symmetrization procedure and, therefore, includes the spin of matter [20]. Note that the normal coordinates y^i form the components of the local radius-vector $\vec{y}$ with respect to the origin $\mathbf{P}$. In consequence, the components of the ${}_mM^{ikl}(y)$ form a tensor density.

For the gravitational field we take the gravitational angular momentum pseudotensor proposed by Bergmann and Thomson [20] as

$$ {}_gM^{ikl}(y) = {}_F U^{i[kl]}(y) - {}_F U^{k[il]}(y) + \sqrt{|g|}(y^i_{BT}t^{kl} - y^k_{BT}t^{il}), \tag{3.5} $$

where

$$ {}_FU^{i[kl]} := g^{im}_F U_m{}^{[kl]} \tag{3.6} $$

are von Freud superpotentials, and

$$ {}_{BT}t^{kl} := g^{ki}_E t_i{}^l + \frac{g^{mk}{}_{,p}}{\sqrt{|g|}} \, {}_F U_m{}^{[lp]} \tag{3.7} $$

is the *Bergmann-Thomson* gravitational energy-momentum pseudotensor. It is closely related to the canonical energy-momentum complex and has better transformational properties than the pseudotensor given by Landau and Lifschitz [10,22]. This is why we apply them here.

One can interpret the Bergmann-Thomson pseudotensor as the sum of the spinorial part

$$ S^{ikl} := {}_F U^{i[kl]} - {}_F U^{k[il]} \tag{3.8} $$

and the orbital part

$$O^{ikl} := \sqrt{|g|}(y^i_{BT}t^{kl} - y^k_{BT}t^{il}) \tag{3.9}$$

of the gravitational angular momentum densities.

Substitution of (3.4) and (3.5) (expanded up to third order) into (3.1) gives the *canonical angular supermomentum tensors* for matter and gravitation, respectively [14],

$$\begin{aligned} {}_mS^{abc}(P; v^l) = 2[&(2v^a v^p + g^{ap})\nabla_p T^{bc} \\ &- (2v^b v^p + g^{bp})\nabla_p T^{ac}], \end{aligned} \tag{3.10}$$

$$\begin{aligned} {}_gS^{abc}(P; v^l) = \alpha(2v^p v^t + g^{pt})[&(g^{ac}g^{br} - g^{bc}g^{ar})\nabla_{(t}R_{pr)} \\ &+ 2g^{ar}\nabla_{(t}R^{(b\ \ c)}_{\ \ p\ \ r)} - 2g^{br}\nabla_{(t}R^{(a\ \ c)}_{\ \ p\ \ r)} \\ &+ \frac{2}{3}g^{bc}(\nabla_r R^{r\ \ a}_{\ (t\ \ p)} - \nabla_{(p}R^a_{t)}) \\ &- \frac{2}{3}g^{ac}(\nabla_r R^{r\ \ b}_{\ (t\ \ p)} - \nabla_{(p}R^b_{t)})]. \end{aligned} \tag{3.11}$$

Both these tensors are antisymmetric in the first two indices $S^{abc} = -S^{bac}$. In vacuum, the gravitational canonical angular supermomentum tensor (3.11) simplifies to

$${}_gS^{abc}(P; v^l) = 2\alpha(2\hat{v}^p\hat{v}^t + \hat{g}^{pt})[\hat{g}^{ar}\nabla_{(p}\hat{R}^{(b\ \ c)}_{\ \ t\ \ r)} - \hat{g}^{br}\nabla_{(p}\hat{R}^{(a\ \ c)}_{\ \ t\ \ r)}]. \tag{3.12}$$

Note that the orbital part $O^{ikl} = \sqrt{|g|}(y^i_{BT}t^{kl} - y^k_{BT}t^{il})$ gives no contribution to ${}_gS^{abc}(P; v^l)$. Only the spinorial part $S^{ikl} =_F U^{i[kl]} -_F U^{k[il]}$ contributes. Also, the canonical angular supermomentum tensor ${}_gS^{abc}(P; v^l)$ and ${}_mS^{abc}(P; v^l)$ of gravitation and matter do not require the introduction of the notion of a radius vector. In special relativity the canonical angular supermomentum tensor for matter, similarly as the canonical superenergy tensor for matter, satisfies trivial conservation laws.

Supermomentum tensors can be decomposed into their tensor (t), vector (v) and axial (a) (totally antisymmetric) parts as follows

$$S^{abc} =^{(t)} S^{abc} +^{(v)} S^{abc} +^{(a)} S^{abc}, \tag{3.13}$$

where

$$ {}^{(v)}S^{abc} := \frac{1}{3}(g^{bc}V^a - g^{ac}V^b), \tag{3.14} $$

$$ {}^{(a)}S^{abc} = S^{[abc]} := \epsilon^{dabc}a_d, \tag{3.15} $$

$$ V^a := S^{ab}{}_b, \quad a^d := -\frac{1}{6}\epsilon^{dabc}S_{abc}. \tag{3.16} $$

The tensorial part of ${}^{(t)}S^{abc}$ of a angular supermomentum tensor can be defined as the difference ${}^{(t)}S^{abc} := S^{abc} - ({}^{(v)}S^{abc} +^{(a)} S^{abc})$.

The canonical superenergy tensors and the canonical angular supermomentum tensors have successfully been calculated for plane, plane-fronted and cylindrical gravitational waves, Friedmann universes, Schwarzschild, and Kerr spacetimes [12–15].

The results are quite promising. For example, the superenergy densities (which are scalars) are positive-definite for Friedman universes and they are very useful to study the nature of the initial singularity in these universes. Moreover, the components of the angular supermomentum tensors for Friedman universes are equal to zero. By use of the our superenergy and angular supermomentum tensors one can also prove that a real gravitational wave with $R_{iklm} \neq 0$ *possesses and carries* positive-definite superenergy, supermomentum and angular supermomentum, and therefore it *must also have and carry* the energy-momentum and the angular momentum. The general conclusion from our investigations in this field is that the canonical superenergy and canonical angular supermomentum tensors give a very useful tool for local and global analysis of the gravitational and matter fields.

IV. OTHER APPROACHES TO SUPERENERGY TENSORS

As we have already remarked we have introduced the superenergy and angular supermomentum tensors for matter owing to universality of these tensors. We think that these tensors are necessary for a complete description of matter and gravity. After introducing the canonical superenergy and angular supermomentum tensors for matter one can define the *total canonical superenergy and angular supermomentum tensors* for matter and grav-

itation [12–15] (as sums of these tensors) and apply these total superenergy and angular supermomentum tensors to analyse a closed system, for example. The total superenergy and angular supermomentum tensors in an obvious way correspond to energy-momentum and angular momentum complexes. The total canonical superenergy and total canonical angular supermomentum tensors allow to study the *exchange* of superenergy and supermomentum between gravitation and matter. Up to now, this problem has just been studied qualitatively.

Of course, there exist other approaches to the problem of superenergy tensors for gravitation and for matter. All of them originated from attempts to interpret physically the Bel-Robinson tensor [23–27]. Recent and the most thorough investigations in this field (restricted to superenergy tensors only) have been given by Senovilla [24].

Senovilla proposes a very general and pure algebraic method for a construction of an infinite sequence of the *super$^{(k)}$–energy tensors* ($k = 1, 2, 3, ...$) for any linear physical field Ψ. This method is independent of the field equations and formalism of the canonical energy-momentum. It is a formal generalization (onto any physical field Ψ which satisfies linear field equations and onto its covariant derivatives of an arbitrary order) of the algebraic method of the construction of the symmetric energy-momentum tensor for electromagnetic field and of the Bel-Robinson and Bel tensors[4]. Of course, a symmetric energy-momentum of the field Ψ, the symmetric energy-momentum tensor for the electromagnetic field, the Bel-Robinson and Bel tensors are included in the infinite sequences of the super$^{(k)}$ –energy tensors. For example, the Bel-Robinson tensor is the super$^{(1)}$ –energy-tensor for the Weyl curvature tensor field and the symmetric energy-momentum tensor for electromagnetic field is super$^{(2)}$ –energy tensor for this field.

A general method of construction of the super$^{(k)}$–energy tensors ($k = 1, 2, 3, ...$) for a linear field (but only in special relativity), similar to the method given in [24], was also

[4]In vacuum these two tensors coincide.

proposed by Teyssandier [25].

It is difficult to find a link between our superenergy tensors and the tensors given in [24] (or in [25]). One can only say that in general our superenergy tensors correspond, in some sense, to those superenergy tensors given in [24] (or in [25]) which are constructed from derivatives of the third order of a physical field Ψ,i.e., *our superenergy tensors correspond to the super*$^{(4)}$*- energy tensors* of Senovilla (or Teyssandier).

Our approach seems to be much simpler than the approach developed in [24] (or in [25]) and it has profound physical meaning: our superenergy (and supermomentum) tensors *are simply the tensors of an averaged relative energy-momentum* (and relative angular momentum). In gravitational case our superenergy and angular supermomentum tensors *extract* covariant information about gravitational field which is hidden in gravitational pseudotensors.

In special relativity our superenergy and supermomentum tensors *do not introduce* any new intrinsic conservation laws for a closed system apart from those which are satisfied by the symmetric energy-momentum tensor. On the other hand, the approach developed in [24] (or in [25]) is far from the standard formalism of the energy-momentum, e.g., in this approach one can try to give a physical meaning to infinite sequence of derivatives of a linear physical field Ψ and in special relativity this approach leads to an infinitely many conservation laws for such a field and its derivatives. From these conservation laws, in fact, only the 10 conservation laws, satisfied by the symmetric energy-momentum tensor of Ψ, can be physically valid.

Our superenergy and angular supermomentum tensors satisfy only 10 local conservation laws in special relativity as a consequence of the 10 conservation laws which are satisfied by the symmetric energy-momentum tensor of matter. However, these local conservation laws *are trivial* in the sense that *they do not lead* to any new integral conservation laws for a closed system because the integral superenergetic quantities calculated from our superenergy and angular supermomentum tensors are all equal to zero for a closed system. So, we have only 10 intrinsic conservation laws which are satisfied by the symmetric energy-momentum

tensor, i.e., exactly the number of conservation laws required in special relativity.

It seems to us that one needs also some deeper physical interpretation to super$^{(k)}$–energy tensors ($k = 1, 2, 3, ...$) (except those which are simply symmetric energy-momentum tensors of the appropriate fields). This is because all the physical content of any physical field Ψ in general relativity is contained in Einstein equations with the symmetric energy-momentum of this field as sources and in the field equations which are satisfied by the field Ψ.

Resuming, we think that our definition of the superenergy (and angular supermomentum) tensors in general relativity is very useful from practical point of view and has a good physical motivation. In consequence, we will confine in the following to our approach.

V. SUPERENERGETIC QUANTITIES FOR ACAUSAL GÖDEL SPACETIME

The Gödel metric describes a space-time homogeneous, but anisotropic universe [1]. There exists a five-dimensional group of isometries which acts in Gödel universe and it is transitive. Its line element in cylindrical coordinates $(x^0, x^1, x^2, x^3) \equiv (t, r, \psi, z)$ in an orthonormal frame is given by [5]

$$ds^2 = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2, \tag{5.1}$$

where

$$\begin{aligned} e^0 &= dt + H(r)d\psi \\ e^1 &= dr \\ e^2 &= D(r)d\psi \\ e^3 &= dz \end{aligned} \tag{5.2}$$

and the radial functions have the form

$$H(r) = \sqrt{2}D(r) = e^{\sqrt{2}\Omega r} \tag{5.3}$$

with Ω constant. In a Gödel universe, the four-velocity of matter is $u^l = \delta^l_0$ and the rotation vector is $V^l = \Omega\delta^l_3$, while the vorticity scalar is given by $\omega = \Omega/\sqrt{2}$. The Gödel metric

(5.1) is a solution to the Einstein's field equations for matter containing dust with constant energy density ϱ and the (negative) cosmological constant. The energy-momentum tensor is [28]

$$T_{ab} = T_{ab}^{(d)} + T_{ab}^{\Lambda}, \tag{5.4}$$

where

$$T_{ab}^{(d)} = \varrho v_a v_b, \tag{5.5}$$

$$T_{ab}^{(\Lambda)} = -\frac{\Lambda}{8\pi}\eta_{ab}, \tag{5.6}$$

or, explicitely,

$$T_{00}^{(d)} = \varrho, \quad T_{11}^{(d)} = T_{22}^{(d)} = T_{33}^{(d)} = 0 \tag{5.7}$$

$$T_{00}^{(\Lambda)} = -T_{11}^{(\Lambda)} = -T_{22}^{(\Lambda)} = -T_{33}^{(\Lambda)} = \frac{\Lambda}{8\pi}, \tag{5.8}$$

and the following relation must be fulfilled

$$4\pi\varrho = \Omega^2 = -\Lambda = \text{const.} \tag{5.9}$$

The unexpected property of the metric (5.1) with H and D given by (5.3) is that it permits time travel, i.e., there exists a closed timelike curve (CTC) through every point of spacetime [2]. In other words, this spacetime contains a closed chronological curve and so a chronology-violating time machine [28]. A time machine is an object or a system which permits travel into the past - this leads to a paradox, since one is then also able to influence one's own future (which is also one's past). There exist chronology-violating time machines and causality-violating time machines (those which allows either timelike or null closed curves). Chronology violation implies causality violation and this is why we speak in this section about an acausal Gödel model. Another problem is that CTCs make it impossible to foliate Gödel spacetime into spacelike hypersurfaces, so that Cauchy problem is ill-posed since one cannot say what are the "initial data" which evolve (in fact, these data do not exist at all). This also means there exists no global cosmic time coordinate t, despite the fact that the Gödel spacetime is geodesically complete.

Since even the simple Minkowski spacetime can be made to contain CTCs by simply identifying points with $t = 0$ and $t = T$ for $t \in [0, T]$ which nobody believes it is acceptable, then most of the physicists consider Gödel model as unphysical [28]. This was best expressed in terms of the Hawking's chronology protection conjecture [29] which says that time travel is completely forbidden in the universe. As we shall see in Section VI one is able to find a generalized Gödel model which is chronology-protected.

Because in acausal Gödel universe the spatial hypersurfaces $t = const$ do not exist we confine only to the local analysis of the superenergy and angular supermomentum in this universe.

The only nonvanishing components of the Ricci rotation coefficients γ^k_{ab} for the metric (5.1) are given in the Appendix while the Riemann tensor components in an orthonormal frame permitted by the spacetime homogeneity of the Gödel universe are [5,6]

$$R_{0101} = R_{0202} = \frac{1}{4}\left(\frac{H'}{D}\right)^2 = \Omega^2, \qquad R_{1212} = \frac{3}{4}\left(\frac{H'}{D}\right)^2 - \frac{D''}{D} = \Omega^2, \tag{5.10}$$

and the prime means the derivative with respect to r. The nonzero components of the Ricci tensor and the Ricci scalar read as

$$R_{00} = 2\Omega^2, \qquad R_{11} = R_{22} = 0, \qquad R = -2\Omega^2. \tag{5.11}$$

In this paper we take the natural orthonormal frame (5.1) for Gödel spacetime as the basic tetrads of the $\mathbf{NC(P)}$ ($\mathbf{P}$ – variable). In consequence, we have $v^l = u^l = \delta^l_0$ and $g^{ab}(\mathbf{P}) = \eta^{ab}$.

First let us calculate the superenergy of matter. From (2.21) we have for the metric (5.1)

$${}_mS_{ab} = T_{ab;(00)} + T_{ab;(11)} + T_{ab;(22)} + T_{ab;(33)}, \tag{5.12}$$

where we have replaced $\nabla(\ldots)$ by $(\ldots)_;$. Since

$$T^{(\Lambda)}_{ma;bc} = (-\Lambda)\eta_{ma;bc} = (-\Lambda)\left\{\left[\eta_{ma,b} - \gamma^k_{mb}\eta_{ka} - \gamma^k_{ab}\eta_{mk}\right]_{,c} - \gamma^k_{mc}\left(\eta_{ka,b} - \gamma^l_{kb}\eta_{la} - \gamma^l_{ab}\eta_{kl}\right)\right.$$
$$\left. -\gamma^k_{ac}\left(\eta_{mk,b} - \gamma^l_{mb}\eta_{lk} - \gamma^l_{kb}\eta_{ml}\right) - \gamma^k_{bc}\left(\eta_{ma,k} - \gamma^l_{mk}\eta_{la} - \gamma^l_{ak}\eta_{ml}\right)\right\}, \tag{5.13}$$

then applying the appropriate coefficients γ^a_{bc} for metric (5.1) (see Appendix) one can easily show that the cosmological constant *does not contribute* to superenergy at all, i.e.,

$$
{}_mS^{(\Lambda)}_{ab} = 0. \tag{5.14}
$$

However, for the dust we have

$$
\begin{aligned}
T^{(d)}_{ma;bc} = {} & \left[T^{(d)}_{ma,b} - \gamma^k_{mb}T^{(d)}_{ka} - \gamma^k_{ab}T^{(d)}_{mk}\right]_{,c} - \gamma^k_{mc}\left(T^{(d)}_{ka,b} - \gamma^l_{kb}T^{(d)}_{la} - \gamma^l_{ab}T^{(d)}_{kl}\right) \\
& -\gamma^k_{ac}\left(T^{(d)}_{mk,b} - \gamma^l_{mb}T^{(d)}_{lk} - \gamma^l_{kb}T^{(d)}_{ml}\right) - \gamma^k_{bc}\left(T^{(d)}_{ma,k} - \gamma^l_{mk}T^{(d)}_{la} - \gamma^l_{ak}T^{(d)}_{ml}\right),
\end{aligned} \tag{5.15}
$$

and we calculate that it contributes to the superenergy of matter, and the canonical superenergy tensor (2.21) for matter fields read as

$$
{}_mS^0_0 = -\frac{\Omega^4}{\pi}, \quad {}_mS^1_1 ={}_m S^2_2 = \frac{\Omega^4}{2\pi}. \tag{5.16}
$$

It is obvious from (5.16) that the superenergy tensor of matter for Gödel models is traceless.

For the gravitational field, the non-vanishing components of the canonical superenergy tensor (2.23) are

$$
{}_gS^0_0 = -\frac{\Omega^4}{36\pi}, \quad {}_gS^1_1 ={}_g S^2_2 = -\frac{\Omega^4}{9\pi}, \quad {}_gS^3_3 = -\frac{7\Omega^4}{36\pi}. \tag{5.17}
$$

From (5.17) we notice that superenergy density S_{00} is positive.

On the other hand, the components of the canonical angular supermomentum tensors for matter fields (3.10) are

$$
{}_mS^{012} = \frac{\Omega^3}{2\pi}, \quad {}_mS^{201} = \frac{\Omega^3}{2\pi}, \quad {}_mS^{120} = -\frac{\Omega^3}{\pi}, \tag{5.18}
$$

while for the gravitational field (3.11) one has

$$
{}_gS^{012} = \frac{4\Omega^3}{24\pi}, \quad {}_gS^{201} = \frac{13\Omega^3}{24\pi}, \quad {}_gS^{120} = -\frac{17\Omega^3}{24\pi}. \tag{5.19}
$$

It is easy to notice that axial (totally antisymmetric) parts of both ${}_mS^{abc}$ and ${}_gS^{abc}$ given by equations (5.18) and (5.19) for Gödel model (5.1) vanish, i.e., the antisymmetric part

$$
{}^{(a)}S^{abc} = S^{[abc]} = \epsilon^{3abc}A_3 = 0, \tag{5.20}
$$

where

$$
A_3 = -\frac{1}{6}\epsilon_{3abc}S^{abc}. \tag{5.21}
$$

This is also the case for the vector part of supermomentum tensor ${}^{(V)}S^{abc} = 0$, so that only pure tensorial part remains non-vanishing (see Eq. (3.13)).

VI. SUPERENERGETIC QUANTITIES FOR CAUSAL GÖDEL SPACETIME

A completely causal Gödel universe with matter source being the scalar field without potential and (negative) cosmological constant has been found by Rebouças and Tiomno [5]. Its metric can be written in the form (5.1) provided we replace $H(r)$ and $D(r)$ into the following generalized functions of the radial coordinate

$$H(r) = \frac{4\Omega}{m^2}\sinh^2\left(\frac{mr}{2}\right), \tag{6.1}$$

$$D(r) = \frac{1}{m}\sinh(mr). \tag{6.2}$$

This allows to write the metric (5.1) in the form

$$ds^2 = -dt^2 - 2H(r)dtd\psi + G(r)d\psi^2 + dr^2 + dz^2, \tag{6.3}$$

where

$$G(r) = \frac{4}{m^2}\sinh^2\left(\frac{mr}{2}\right)\left[1 + \left(1 - \frac{4\Omega^2}{m^2}\right)\sinh^2\left(\frac{mr}{2}\right)\right], \tag{6.4}$$

with m and Ω constants. This model is causal (there are no CTCs) because

$$G(r) = D^2(r) - H^2(r) > 0 \tag{6.5}$$

for

$$4\Omega^2 = m^2, \tag{6.6}$$

and the term in front of $d\psi^2$ in the metric (6.3) remains positive. The conditions (6.5) and (6.6) remove CTCs to a point which is formally at $r \to \infty$. On the other hand, the function $H(r)$ can always be made zero in a more general class of models studied in Refs. [30,31] for which there are no CTCs for any value of the radial coordinate $r > 0$. The Gödel model (5.1) is obtained by taking

$$2\Omega^2 = m^2 \tag{6.7}$$

in (5.3) and it does not fulfil the condition (6.5).

The only nonvanishing components of the Riemann tensor for the metric (6.3) in an orthonormal frame are [5,6]

$$R_{0101} = R_{0202} = \frac{1}{4}\left(\frac{H'}{D}\right)^2 = \Omega^2, \quad R_{1212} = \frac{3}{4}\left(\frac{H'}{D}\right)^2 - \frac{D''}{D} = -\Omega^2. \tag{6.8}$$

The nonzero components of the Ricci tensor and the Ricci scalar read as

$$R_{00} = 2\Omega^2, \quad R_{11} = R_{22} = -2\Omega^2, \quad R = -6\Omega^2. \tag{6.9}$$

The scalar field depends only on the coordinate along the axis of rotation, z, i.e.,

$$\phi = \phi(z) = ez + \phi_0, \tag{6.10}$$

where e and ϕ_0 are constants. The energy-momentum tensor of the scalar field reads as

$$T^{(\phi)}_{ab} = \phi_{;a}\phi_{;b} - \frac{1}{2}\eta_{ab}\phi_{;m}\phi^{;m}, \tag{6.11}$$

so because $\phi_{;a} = e\delta^3_a$, then one has

$$T_{00} = -T_{11} = -T_{22} = T_{33} = \frac{1}{2}e^2. \tag{6.12}$$

For the cosmological term we have the energy-momentum tensor given again by (5.6). The following relation between parameters Λ, Ω and e has to be fulfilled

$$\Lambda = -2\Omega^2 = -8\pi e^2. \tag{6.13}$$

After simple, but tedious calculations (see Appendix) we have found that all components of the canonical superenergy tensor (2.21) for matter fields (6.11)-(5.6) vanish (we have already shown in the Section V that this is the case for the cosmological constant), i.e.

$$_mS_a^b = 0 \tag{6.14}$$

For the gravitational field, the non-vanishing components of the canonical superenergy tensor (2.23) are

$$_gS_0^0 = \frac{\Omega^4}{4\pi}, \quad _gS_1^1 =_g S_2^2 = -\frac{\Omega^4}{36\pi}, \quad _gS_3^3 = -\frac{\Omega^4}{12\pi}. \tag{6.15}$$

From (6.15) it follows that the price to pay for causality is the negative superenergy density ${}_gS_{00} < 0$.

On the other hand, the components of the canonical angular supermomentum tensors for matter fields (3.10) vanish

$$ {}_mS^{abc} = 0, \tag{6.16} $$

while for the gravitational field (3.11) one has

$$ {}_gS^{012} = 0, \quad {}_gS^{201} = \frac{\sqrt{2}\Omega^3}{24\pi}, \quad {}_gS^{120} = -\frac{\sqrt{2}\Omega^3}{24\pi}. \tag{6.17} $$

It is easy to notice that ${}_gS^{abc}$ given by (6.17) for Gödel model (6.3) is axial-free and vector-free (cf. Eq. (3.13), i.e.,

$$ {}^{(a)}S^{[abc]} = \epsilon^{3abc}A_3 = 0, \tag{6.18} $$

$$ {}^{(V)}S^{abc} = 0. \tag{6.19} $$

VII. CONCLUSION

We have calculated the canonical superenergy tensor and the canonical angular supermomentum tensor for homogeneous rotating Gödel universes. We considered the original pressureless dust plus dosmological constant model found by Gödel in 1949 which admits CTCs (acausal model) and the scalar-field plus cosmological constant model found by Rebouças and Tiomno in 1983 which is free from CTCs (causal model). Due to the peculiarity of Gödel spacetimes we expected the appearance of some interesting properties of the calculated superenergetic quantities. On the other hand, because of rotation we expected the appearance of some non-vanishing components of the angular supermomentum tensor. For the acausal model the non-vanishing components of superenergy of matter are different from those of gravitation. The matter superenergy tensor is traceless. The angular supermomentum tensors of matter and gravitation do not vanish either which simply reflects the fact

that Gödel universe rotates. It emerges that the supermomentum tensors have vanishing totally antisymmetric (axial) and vectorial parts. For the causal model superenergy and supermomentum of matter vanish. However, superenergy and supermomentum of gravitation do not vanish. On the other hand, superenergy density for the causal model is negative and its supermomentum is axial-free and vector-free as for the acausal model.

It is interesting that superenergetic quantities are *sensitive* to causality in a way that superenergy density ${}_gS_{00}$ of gravitation in the acausal model is *positive*, while superenergy density ${}_gS_{00}$ in the causal model is *negative*. That means superenergetic quantities might serve as criterion of causality in cosmology and prove useful.

Another point is that although both Gödel-type models have as a source of gravity the cosmological constant it does not contribute to their superenergy and supermomentum at all.

For our requirements it is important that in the acausal Gödel universe does not exist a global cosmic time coordinate t. This is the reason why we confine only to local analysis of the superenergy and the angular supermomentum in this universe. However, it is different in a causal Gödel-Rebouças-Tiomno universe where the global hypersurfaces $t = const.$ exist, so it would be possible to calculate the energetic quantities, i.e., the energy, the momentum and the angular momentum for these universes and compare the results with superenergetic quantities. Such analysis has already been given for Friedman universes [13,14].

Finally, we believe that superenergetic quantities give quite a lot of physical information about the specific properties of Gödel spacetime which make these quantities useful.

VIII. ACKNOWLEDGMENTS

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APPENDIX A: USEFUL FORMULAS TO CALCULATE SUPERENERGY AND SUPERMOMENTUM

The Ricci rotation coefficients for a generalized Gödel model (6.3) are

$$\begin{aligned}\gamma^0_{12} = \gamma^1_{20} = \gamma^1_{02} &= \frac{m}{\sqrt{2}},\\ \gamma^0_{21} = \gamma^2_{10} = \gamma^2_{01} &= -\frac{m}{\sqrt{2}},\\ \gamma^1_{22} = -\gamma^2_{12} &= -m.\end{aligned} \tag{A1}$$

According to (6.6) and (6.7) one has to put $m = \sqrt{2}\Omega$ for Gödel model and $m = 2\Omega$ for Rebouças and Tiomno model.

The nonzero components of the Bel-Robinson tensor (2.25) are

$$\begin{aligned}&{B^0}_{00}{}^0 = -{B^3}_{30}{}^0 = 2\Omega^4,\\ &{B^0}_{01}{}^1 = {B^0}_{02}{}^2 = {B^0}_{03}{}^3 = -{B^1}_{12}{}^2 = -{B^2}_{21}{}^1 = -8\Omega^4 + 6\Omega^2 m^2 - m^4,\\ &{B^2}_{22}{}^2 = {B^1}_{11}{}^1 = -{B^3}_{31}{}^1 = -{B^3}_{32}{}^2 = 10\Omega^4 - 6\Omega^2 m^2 + m^4.\end{aligned} \tag{A2}$$

The non-vanishing components of the tensor (2.26) are

$$\begin{aligned}&{P^0}_{00}{}^0 = 3\Omega^4, \qquad {P^1}_{10}{}^0 = {P^2}_{20}{}^0 = -{P^3}_{30}{}^0 = -{P^3}_{31}{}^1 = -{P^3}_{32}{}^2 = \Omega^4,\\ &{P^0}_{01}{}^1 = {P^0}_{02}{}^2 = {P^0}_{03}{}^3 = -\frac{1}{2}\left(6\Omega^4 - 6\Omega^2 m^2 + m^4\right),\\ &{P^1}_{11}{}^1 = {P^2}_{22}{}^2 = \frac{3}{2}\left(10\Omega^4 - 6\Omega^2 m^2 + m^4\right),\\ &{P^1}_{12}{}^2 = {P^2}_{21}{}^1 = \frac{1}{2}\left(26\Omega^4 - 18\Omega^2 m^2 + m^4\right).\end{aligned} \tag{A3}$$

We also define (see Eq. (2.24))

$$Riem2^l_l := -\frac{1}{2}{R^{ijk}}_m(R_{ijkl} + R_{ikjl}), \tag{A4}$$

which has non-vanishing components

$$Riem2^0_0 = -3\Omega^4, \qquad Riem2^1_1 = Riem2^2_2 = -\frac{3}{2}\left(10\Omega^4 - 6\Omega^2 m^2 + m^4\right). \tag{A5}$$

Another object which we define is

$$ {}^{a}_{a}Ric_{ll} := 2\delta^b_a R_{(l|g} R^{g}{}_{|l)} - 3R_{a(l|} R^{b}{}_{|l)} + 2R^{b}{}_{(ag)(l|} R^{g}{}_{|l)}, \tag{A6} $$

and its nonzero components are

$$ \begin{aligned} &{}^0_0Ric^0_0 = -4\Omega^4, \qquad {}^1_1Ric^0_0 ={}^2_2\, Ric^0_0 = 10\Omega^4, \qquad {}^3_3Ric^0_0 = 8\Omega^4, \\ &{}^0_0Ric^1_1 = {}^0_0Ric^2_2 = 6\Omega^4 - 7\Omega^2 m^2 + 2m^4, \qquad {}^1_1Ric^2_2 ={}^2_2\, Ric^1_1 = 14\Omega^4 - 13\Omega^2 m^2 + 3m^4, \\ &{}^1_1Ric^1_1 = {}^2_2Ric^2_2 = -4\Omega^4 + 4\Omega^2 m^2 - m^4, \qquad {}^3_3Ric^1_1 ={}^3_3\, Ric^2_2 = \left(2\Omega^2 - m^2\right)^2. \end{aligned} \tag{A7} $$

Finally, the non-vanishing components of the superenergy tensor (2.23) are

$$ \begin{aligned} {}_gS^1_1 = {}_gS^2_2 &= \frac{2\alpha}{9}\left[18\Omega^4 - 21\Omega^2 m^2 + 4m^4\right], \\ gS^3_3 &= \frac{2\alpha}{9}\left[-30\Omega^4 + 10\Omega^2 m^2 - m^4\right], \\ gS^0_0 &= \frac{2\alpha}{9}\left[-38\Omega^4 + 22\Omega^2 m^2 - 2m^4\right]. \end{aligned} \tag{A8} $$

In order to calculate supermomentum of gravity we need covariant derivatives of the Riemann tensor

$$ R^{b}{}_{p}{}^{c}{}_{r;t} = R^{b}{}_{p}{}^{c}{}_{r,t} + \gamma^b_{kt} R^{k}{}_{p}{}^{c}{}_{r} - \gamma^k_{pt} R^{b}{}_{k}{}^{c}{}_{r} + \gamma^c_{kt} R^{b}{}_{p}{}^{k}{}_{r} - \gamma^k_{rt} R^{b}{}_{p}{}^{c}{}_{k}, \tag{A9} $$

which have nonzero components

$$ \begin{aligned} &R^{2}{}_{0}{}^{1}{}_{0;0} = R^{1}{}_{0}{}^{2}{}_{0;0} = \frac{m}{\sqrt{2}}\Omega^2, \qquad R^{2}{}_{1}{}^{1}{}_{0;1} = R^{1}{}_{0}{}^{2}{}_{1;1} = -\frac{m}{\sqrt{2}}\left(-2\Omega^2 + m^2\right), \\ &R^{1}{}_{2}{}^{2}{}_{0;2} = R^{2}{}_{0}{}^{1}{}_{2;2} = R^{0}{}_{2}{}^{1}{}_{2;2} = R^{1}{}_{2}{}^{0}{}_{2;2} = R^{0}{}_{1}{}^{1}{}_{2;1} = R^{1}{}_{2}{}^{0}{}_{1;1} = \frac{m}{\sqrt{2}}\left(4\Omega^2 - m^2\right), \\ &R^{0}{}_{2}{}^{2}{}_{1;2} = R^{2}{}_{1}{}^{0}{}_{2;2} = R^{0}{}_{1}{}^{2}{}_{1;1} = R^{2}{}_{1}{}^{0}{}_{1;1} = -\frac{m}{\sqrt{2}}\left(4\Omega^2 - m^2\right), \end{aligned} \tag{A10} $$

and finally for (3.11) we have

$$ \begin{aligned} \frac{1}{\alpha}{}_gS^{012} &= \frac{2\sqrt{2}m}{3}\left(4\Omega^2 - m^2\right), \\ \frac{1}{\alpha}{}_gS^{021} &= \frac{\sqrt{2}m}{3}\left(-25\Omega^2 + 6m^2\right), \\ \frac{1}{\alpha}{}_gS^{120} &= \frac{\sqrt{2}m}{3}\left(-33\Omega^2 + 8m^2\right). \end{aligned} \tag{A11} $$

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